Homework. Ecole Polytechnique. MAP670R-2022 Advanced Topics in Deep Learning.

Fix $d \in \mathbb{N}^*$. We introduce $L^2(\mathbb{R}^d) = \{f \text{ measurable}, \int_{\mathbb{R}^d} |f(x)|^2 \, dx < \infty\}$ and the action of smooth diffeomorphisms $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$ given for $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ by:

$$L_\phi f(x) \triangleq f(\phi^{-1}(x)) \,.$$

For an operator $\tilde{M} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, we say that $\tilde{M}$ is Lipschitz (or bounded, if linear) if $\|\tilde{M}\| \triangleq \sup_{f \neq g} \frac{\|\tilde{M}f - \tilde{M}g\|}{\|f-g\|} < \infty$. The goal of this homework is to show that the Lipschitz operators (potentially non-linear) which commute with any bounded diffeomorphism action are the Lipschitz pointwise non-linearity which vanish in 0. In otherwords, we study operators $M : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that for any $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ diffeomorphisms such that $\|L_\phi\| < \infty$, we get:

$$ML_\phi = L_\phi M \,.$$

For a diffeomorphism ϕ, we write its support $\mathcal{S}(\phi) \triangleq \overline{\{x, \phi(x) \neq x\}}$. We say ϕ has compact support if $\mathcal{S}(\phi)$ is compact. We write $\mathcal{B}(x, \rho) = \{y, \|x - y\| < \rho\} \subset \mathbb{R}^d$ a real ball centered in $x \in \mathbb{R}^d$ and of radii $\rho > 0$, $\bar{A}$ is the closure of A and $\partial\tau$ is the differential of $\tau \in \mathcal{C}^\infty(\mathbb{R}^d)$.

Exercise 1 (Action of diffeomorphisms on $L^2(\mathbb{R}^d)$.) We focus on several preliminary properties to prove the main result of this homework.

1. Show that if a smooth diffeomorphism ϕ has a compact support, then L_ϕ is a bounded operator of $L^2(\mathbb{R}^d)$.
2. Show that if $\phi(x) = Ax + b$ for A an invertible matrix and $b \in \mathbb{R}^d$, then ϕ defines a bounded operator.
3. Show that if $\phi(x) = x - \tau(x)$ with $\tau \in \mathcal{C}^\infty$ such that $\sup_{x \in \mathbb{R}^d} \|\partial\tau(x)\| \leq \frac{1}{2}$, then ϕ is a smooth diffeomorphism and L_ϕ is a bounded operator.
4. Let $\rho > 0, x \in \mathbb{R}^d$, show that for any $x_0, x_1 \in \mathcal{B}(x, \rho)$, there exists a smooth diffeomorphism ϕ_{x_0,x_1} such that $\mathcal{S}(\phi_{x_0,x_1}) \subset \mathcal{B}(x, \rho)$ and $\phi_{x_0,x_1}(x_0) = x_1$. **Hint:** Use a connexity argument.
5. For any $\epsilon > 0, n \in \mathbb{N}^*$, show that there exists an increasing smooth function f_n such that $f_n(1) = \frac{1}{n}$ and $f_n(1 + \epsilon) = 1$. Deduce that for any balls $\overline{\mathcal{B}} \subset \mathcal{B}'$, there exists a sequence of bounded diffeomorphisms ϕ_n supported in $\mathcal{B}'$ such that $\|L_{\phi_n} 1_{\mathcal{B}}\| \to 0$.

Exercise 2 (Action of M on $\mathbb{R}^d$-balls.) Assume that M is a Lipschitz operator which commutes with the action of diffeomorphisms.

1. Show that $M(\mathbf{0}) = \mathbf{0}$.
2. Fix $(\alpha, x, \rho) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+^*$. Using Exercise 1, Question 4, show that $M(\alpha 1_{\mathcal{B}(x,\rho)}) = F(\alpha, x, \rho) 1_{\mathcal{B}(x,\rho)}$ with $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{R}$.
3. Show that there exists $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ such that for any $(\alpha, x, \rho) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+^*$, we have:
$$F(\alpha, x, \rho) = \mathfrak{h}(\alpha) \,.$$
4. Show the conclusion also holds almost surely for the closed ball $\overline{\mathcal{B}(x, \rho)}$, i.e. that:
$$M(\alpha 1_{\overline{\mathcal{B}(x,\rho)}}) = \mathfrak{h}(\alpha) 1_{\overline{\mathcal{B}(x,\rho)}} \,.$$
5. Prove that $\mathfrak{h}(0) = 0$ and that $\mathfrak{h}$ is $\|M\|$-Lipschitz.

Exercise 3 (Action of M on $L^2(\mathbb{R}^d)$.) The goal is to extend the previous results to arbitrary functions of $L^2(\mathbb{R}^d)$. We admit the Vitali's Lemma which states that for any $\epsilon > 0$ and $f \in L^2(\mathbb{R}^d)$, there exists $n \in \mathbb{N}^*$, $(\alpha_1, x_1, \rho_1), ..., (\alpha_n, x_n, \rho_n)$ such that $i \neq j \Rightarrow \overline{\mathcal{B}(x_i, \rho_i)} \cap \overline{\mathcal{B}(x_j, \rho_j)} = \emptyset$ and:

$$\|\sum_{i=1}^n \alpha_i 1_{\overline{\mathcal{B}(x_i,\rho_i)}} - f\| \leq \epsilon \,.$$

1. Assume that $i \neq j \Rightarrow \overline{\mathcal{B}(x_i, \rho_i)} \cap \overline{\mathcal{B}(x_j, \rho_j)} = \emptyset$ and let $(\alpha_i, x_i, \rho_i)_{i \leq n}$. Using Exercise 1, Question 5, prove by induction that:

$$M(\sum_{i=1}^{n} \alpha_i 1_{\overline{\mathcal{B}(x_i, \rho_i)}}) = \sum_{i=1}^{n} M(\alpha_i 1_{\overline{\mathcal{B}(x_i, \rho_i)}}) \text{ a.s. } .$$

2. Show that for any $f \in L^2(\mathbb{R}^d)$:

$$\forall x \in \mathbb{R}^d, Mf(x) = \mathfrak{h}(f(x)) \text{ a.s. } ,$$

where $\mathfrak{h}$ is as in the Exercise 2.

3. Conclude.